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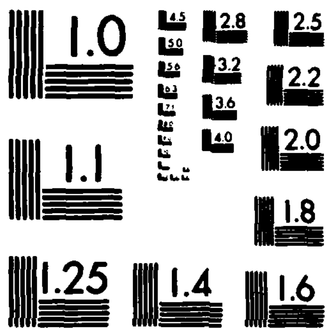
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A WELL-POSEDNESS PROPERTY OF A CLASS OF VARIATIONAL PROBLEMS AND ITS APPLICATION TO NONLINEAR ESTIMATION

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ABSTRACT

In this paper we consider a class of nonlinear estimators that are the solutions to a certain variational problem. These estimators generalize the concept of conditional expectation; we investigate their continuity and convergence properties.

I. INTRODUCTION

In this paper we will develop some machinery that generalizes some well known approximation and continuity properties of minimum mean square error analysis. These will allow us to examine estimation theory in a much more general context and give us a much wider choice of criteria for analyzing and penalizing error. We will be required to forego some of the main convenient accoutrements of minimum mean square error analysis: the Hilbert space structure of L^2, the projection theorem, and the fact that minimum norm projection onto a closed subspace is a nonexpansive linear operator. However, in return we receive a method that allows us to customize our notion of error to our particular model. Oftentimes, mean square error is not the appropriate fidelity criterion.

II. PRELIMINARIES

Throughout this paper let $(\Omega,\mathcal{F},P)$ denote a probability space and let $L^0(\Omega,\mathcal{F},P)$ denote the set of all random variables $X:\Omega \to \mathbb{R}$ modulo a.s. equivalence equipped with the topology of convergence in probability.

Let Φ: $[0,\infty) \to [0,\infty)$ be convex, increasing, and satisfy $\Phi(0)=0$. We define the Orlicz space

$$L^\Phi(\Omega,\mathcal{F},P) = \{X\epsilon L^0(\Omega,\mathcal{F},P): \int_\Omega \Phi(|X|)dP < \infty\}.$$

The function Φ is said to satisfy the doubling condition if there exist $C,M > 0$ such that $x \geq M \Rightarrow \Phi(2x) \leq C\Phi(x)$. If Φ satisfies the doubling condition, $L^\Phi(\Omega,\mathcal{F},P)$ is a vector space. Moreover, it is a Banach space with the Luxemburg norm

$$||X||_{L^\Phi} = \inf \{\lambda > 0: \int_\Omega \Phi(\frac{|X|}{\lambda})dP \leq \Phi(1)\}.$$

Note that this is simply the Minkowski functional of a certain subset of L^Φ. Also, if $\{X_n\}_{n=1}^\infty$ and X are in L^Φ, $X_n \xrightarrow{L^\Phi} X$ if and only if $\lim_{n\to\infty} \int_\Omega \Phi(|X_n - X|)dP = 0$.

For the basic facts on Orlicz spaces see [3]. Details on Minkowski functions can be found in [1], pp. 294-295 or [6], pp. 23-26.

Let Φ: $[0,\infty) \to [0,\infty)$ be increasing and have a strictly increasing first derivative ϕ with $\phi(x) \to \infty$ as $x \to \infty$. Since ϕ: $[0,\infty) \to [0,\infty)$ is a homeomor-

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phism, $\psi = \phi^{-1}$ is also continuous, strictly increasing, and satisfies $\lim_{x\to\infty} \psi(x) = \infty$. We define the <u>conjugate function</u> Ψ to Φ to be

$$\Psi(x) = \int_0^x \psi(t)dt, \quad x \geq 0.$$

This is somewhat more restrictive than the definition employed in [3], but it eliminates certain annoying technicalities. It also allows us to dispense with the customary nonatomicity assumption made about the measure spaces used in this context. We call the pair Φ,Ψ an <u>Orlicz pair</u>.

The Banach space L^Φ is reflexive if and only if Ψ and Φ satisfy the doubling condition. A simple calculation shows that the strict convexity of Φ implies that of the Luxemburg norm on L^Φ and that of the convex functional $X \to \int_\Omega \Phi(|X|)dP$. In addition, if the Banach space L^Φ is reflexive, we have $L^{\Phi*} \cong L^\Psi$ [3]. Hereafter we posit all Orlicz spaces are reflexive.

Let B be a Banach space. We say B is (i) <u>locally uniformly convex</u> if whenever $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ are sequences in the unit ball of B with $\lim_{n\to\infty} ||x_n+y_n|| = 2$, $\lim_{n\to\infty}||x_n-y_n|| = 0$; (ii) <u>uniformly convex</u> if $\forall\varepsilon > 0$ there exists $\delta > 0$ so that for any $x,y\epsilon B$ so that $||x|| = ||y|| = 1$, $||x+y|| \geq 2(1-\delta) \Rightarrow ||x-y|| < \varepsilon$. Uniform convexity is a much stronger condition than local uniform convexity. Every uniformly convex space is reflexive [6], pp. 126-128. For more about these sorts of conditions, see [2].

If B is a Banach space, $Q \subseteq B$ is said to be <u>proximal</u> if $\forall x\epsilon B$, the problem

$$\min_{z\epsilon Q} ||x-z||$$

possesses a solution. If B is reflexive we may apply the Smul'lyan theorems to see that every closed convex subset is proximal. Moreover, if B is strictly convex the norm minimizer problem has at most one solution. In this case for $x\epsilon B$ and $K \subseteq B$ that is closed and convex let $p_K(x)$ denote the unique solution to

$$\min_{z\epsilon K} ||x-z||.$$

The map p_K is called the <u>metric projection</u> of B onto K. Shortly we will see that if B is locally uniformly convex and reflexive then for any closed convex $K \subseteq B$, p_K is norm continuous. In [5] we showed that p_K is weakly sequentially continuous if B is reflexive and strictly convex.

Let M be a separable metric space. A Borel measurable map $Q: M \to M$ is said to be a <u>round off map</u> if Q has finite range, say $\{p_1,\ldots,p_n\}$ = range Q and $Q(p_k) = p_k$, $1 \leq k \leq n$. The set $\{Q^{-1}(p_1),\ldots,Q^{-1}(p_n)\}$ is called the <u>partition of M defined by Q</u>. A sequence $\{Q_n\}_{n=1}^\infty$ of round off maps is called a <u>round off scheme</u> if (i) $\forall x\epsilon M$, $\lim_{n\to\infty} \text{dia } Q_n^{-1}(Q_n(x)) = 0$; (ii) the partition of M defined by Q_{n+1} refines that defined by Q_n, $n\epsilon\mathbb{N}$. Note $\sigma(Q_n) \subseteq \sigma(Q_{n+1})$, $n\epsilon\mathbb{N}$. See [5] for details on round off schemes.

The primary result of [5] is contained in the

<u>Theorem 1</u>: Let Φ,Ψ be an Orlicz pair, $(\Omega,\mathscr{S},P)$ be a nonatomic probability space, M be a separable metric space and $\{X(t): t\epsilon[0,T]\}$ be a stochastically continuous process on $(\Omega,\mathscr{S},P)$ taking values in M. Then for any round off scheme $\{Q_n\}_{n=1}^\infty$ on M and any increasing sequence $\{P_m\}_{m=1}^\infty$ of partitions of $[0,T]$ whose meshes decrease to zero we have for any $Y \epsilon L^\Phi(\Omega,\mathscr{S},P)$

$$\| E_\Phi(Y|Q_n(X(t)): t\epsilon P_m) - E_\Phi(Y|X(t): t\epsilon[0,T])\|_{L^\Phi} \to 0$$

as $m,n \to \infty$; or, equivalently,

$$0 = \lim_{m,n\to\infty} \int_\Omega \Phi(|E_\Phi(Y|Q_n(X(t)): t\epsilon P_m) - E_\Phi(Y|X(t): t\epsilon[0,T])|)dP$$

where $E_\Phi(\cdot|\mathscr{F}): L^\Phi(\Omega,\mathscr{S},P) \to L^\Phi(\Omega,\mathscr{F},P)$ denotes metric projection and $\mathscr{F}$ is any σ-subalgebra of $\mathscr{S}$.

III. A CONVEXITY CONDITION FOR CONVEX FUNCTIONALS ON REFLEXIVE BANACH SPACES

Let B be a reflexive Banach space and ψ: $B \to \mathbb{R}$ be a convex, norm continuous map satisfying

$$\lim_{\|x\|\to\infty} \psi(x) = +\infty .$$

Then using the Smul'lyan theorems we see that $\forall x\epsilon B$

$$\min_{z\epsilon K} \psi(x-z) \tag{1}$$

possesses a closed convex nonvoid subset L_x of K of solutions. Furthermore, if ψ is strictly convex each L_x is a singleton $\{p(x)\}$. We call the map p: $B \to K$ the <u>solution map</u> of (1). Let $\delta > 0$; define

$$K_{\delta,x} = \{z\epsilon K: \psi(x-z) < \delta + \psi(x-p(x))\}.$$

We say ψ is <u>locally uniformly convex</u> if $\lim_{\delta\downarrow 0} \text{dia } K_{\delta,x} = 0$ for all $x\epsilon B$ and closed convex $K \subseteq B$.

<u>Proposition 2</u>: If ψ: $B \to \mathbb{R}$ is locally uniformly convex and uniformly continuous on bounded subsets of B, $K \subseteq B$ is nonvoid convex and closed and p: $B \to K$ is the solution map of (1) then p is norm continuous.

<u>Proof</u>: Let $x_n \to x$ in B and $\delta > 0$. Take $z\epsilon K_{\delta,x}$. Then

$$\psi(x-z) < \delta + \psi(x-p(x)).$$

Since $x_n \to x$ there exists $N\epsilon\mathbb{N}$ so that $n \geq N \Rightarrow$

$$\psi(x_n-z) < \delta+\psi(x_n-p(x)).$$

By minimality

$$\begin{aligned}\psi(x_n-p(x_n)) &\leq \psi(x_n-z)\\ &< \delta + \psi(x_n-p(x)).\end{aligned}$$

Taking limit suprema,

$$\limsup_{n\to\infty} \psi(x_n-p(x_n)) \leq \psi(x-p(x)). \tag{2}$$

Suppose by way of the contrapositive that $\{p(x_n)\}_{n=1}^\infty$ is unbounded. Since $x_n \to x$, $\{x_n\}_{n=1}^\infty$ is bounded; this forces $\{x_n-p(x_n)\}_{n=1}^\infty$ to be unbounded. Observing that $\lim_{\|x\|\to\infty} \psi(x) = +\infty$ it now follows that

$$\limsup_{n\to\infty} \psi(x_n-p(x_n)) = +\infty,$$

a flagrant violation of (2). This insures that $\{p(x_n)\}_{n=1}^\infty$ is bounded.

Suppose that for some $z_0\epsilon B$, $p(x_n) \rightharpoonup z_0$. Then $x_n - p(x_n) \rightharpoonup x-z_0$; ψ is convex and norm continuous and therefore weakly lowersemicontinuous. Thus

$$\psi(x-z_0) \leq \liminf_{n\to\infty} \psi(x_n-p(x_n))$$
$$\leq \limsup_{n\to\infty} \psi(x_n-p(x_n)) \leq \psi(x-p(x)).$$

Next observe that K is closed and convex and therefore weakly closed. Thus $p(x_n)\epsilon K$, $n\epsilon\mathbb{N}$, $p(x_n)\rightharpoonup z_0 \Rightarrow z_0\epsilon K$. By minimality, $z_0 = p(x)$.

Since B is reflexive and $\{p(x_n)\}_{n=1}^{\infty}$ is bounded each subsequence of $\{p(x_n)\}_{n=1}^{\infty}$ must have a further <u>weakly</u> <u>convergent</u> subsequence. But the foregoing discussion tells us that this further subsequence must converge to $p(x)$. Immediately it follows $p(x_n) \rightharpoonup p(x)$.

We may apply the weak lower semicontinuity of ψ to see that

$$\psi(x-p(x)) \leq \liminf_{n\to\infty} \psi(x_n-p(x_n)).$$

Combine this with (2) to get

$$\psi(x-p(x)) = \lim_{n\to\infty} \psi(x_n-p(x_n)).$$

Put $M = 1 + \sup_n ||x_n-p(x_n)||$; note $M<\infty$. Take $\varepsilon>0$, choose $\eta>0$ s.t. $\eta<1$ and $||x-y|| <\eta$, $||x||,||y|| \leq M \Rightarrow |\psi(x)-\psi(y)| <\varepsilon$. Because $x_n \to x$ there exists $N\epsilon\mathbb{N}$ such that $n \geq N \Rightarrow ||x_n-x|| < \eta$. Then $n \geq N \Rightarrow$

$$||x_n-p(x_n)-(x-p(x_n))|| = ||x_n-x|| < \eta$$

so $n \geq N \Rightarrow$

$$|\psi(x_n-p(x_n)) - \psi(x-p(x))| < \varepsilon.$$

This forces $\lim_{n\to\infty}\{\psi(x_n-p(x_n))-\psi(x-p(x_n))\} = 0$ and hence $\lim_{n\to\infty} \psi(x-p(x_n)) = \psi(x-p(x))$.

Noting $\lim_{n\to\infty} \psi(x-p(x_n)) = \psi(x-p(x))$ there exists $N\epsilon\mathbb{N}$ so that $n \geq N \Rightarrow$

$$\psi(x-p(x_n)) < \delta + \psi(x-p(x)).$$

But this says that for $n \geq N$, $p(x_n) \epsilon K_{\delta,x}$. Obviously $p(x) \epsilon K_{\delta,x}$ so for $n \geq N$, $||p(x_n)-p(x)|| \leq \text{dia } K_{\delta,x}$. Since $\lim_{\delta\to 0} \text{dia } K_{\delta,x} = 0$ we must have $p(x_n) \to p(x)$. <u>QED</u>

<u>Proposition 3</u>: Suppose $\{K_n\}_{n=1}^{\infty}$, K are nonvoid closed convex subsets of B with $K_n \subseteq K_{n+1}$, $n\epsilon\mathbb{N}$ and $K = \overline{\bigcup_{n=1}^{\infty} K_n}$. Let $\psi: B \to \mathbb{R}$ be locally uniformly convex and norm continuous. Finally denote by p_n the solution map of (1) corresponding to K_n, $n\epsilon\mathbb{N}$ and p denote the solution map of K. Then for $x\epsilon B$, $p_n(x) \to p(x)$.

<u>Proof</u>: Let $\delta>0$, $x\epsilon B$. By the <u>continuity</u> of ψ, $K_{\delta,x}$ is a relative neighborhood of $p(x)$ in K. Noting $K = \overline{\bigcup_{n=1}^{\infty} K_n}$, we see there must be an $N\epsilon\mathbb{N}$ so that $n \geq N \Rightarrow K_n \cap K_{\delta,x} \neq \emptyset$.

Fix $n \geq N$, $z\epsilon K_{\delta,x} \cap K_n$. Then we know that (i) $\psi(x-z) < \delta + \psi(x-p(x))$; (ii) $\psi(x-p_n(x)) \leq \psi(x-z)$. Assembling the facts, $n \geq N \Rightarrow$

$$\psi(x-p_n(x)) \leq \delta + \psi(x-p(x)).$$

Letting $n\to\infty$,

$$\limsup_{n\to\infty} \psi(x-p_n(x)) \le \psi(x-p(x)). \tag{3}$$

We may imitate a familiar argument in Proposition 2 to see that $\{p_n(x)\}_{n=1}^{\infty}$ is bounded.

Since $K_n \subseteq K_{n+1} \subseteq K$, $n\in\mathbb{N}$ it is easy to see that

$$\psi(x-p_n(x)) \ge \psi(x-p_{n+1}(x)) \ge \psi(x-p(x)).$$

With the assistance of (3) we obtain

$$\psi(x-p(x)) = \lim_{n\to\infty} \psi(x-p_n(x)).$$

Pick $\delta > 0$. There exists $N\in\mathbb{N}$ such that $n \ge N \Rightarrow$

$$\psi(x-p_n(x)) < \psi(x-p(x)) + \delta.$$

We may now conclude $p_n(x)\in K_{\delta,x}$, $n \ge N$. Since dia $K_{\delta,x} \to 0$ as $\delta \to 0$ we conclude that $p_n(x) \to p(x)$ in norm. QED

IV. SOME FACTS ABOUT ORLICZ SPACES

Theorem 4: Let $(\Omega,\mathscr{F},P)$ be a probability space and Φ,Ψ be an Orlicz pair. Then $L^{\Phi}(\Omega,\mathscr{F},P)$ is uniformly convex if and only if

$\forall\ \varepsilon$ such that $0 < \varepsilon < \frac{1}{4}$ there exists $R_\varepsilon > 0$ so that

$$\liminf_{n\to\infty} \frac{\psi(x)}{\psi((1-\varepsilon)x)} > R_\varepsilon.$$

Proof: This is the central result of [4].

Define $\xi: L^{\Phi}(\Omega,\mathscr{F},P) \to [0,\infty)$ by $\xi(X) = \int_\Omega \Phi(|X|)dP$. Then ξ is a strictly convex norm continuous functional and $\lim_{||X||_{L^\Phi} \to \infty} \xi(X) = +\infty$. Let $0 < \varepsilon < 1$, $X,Y\in L(\Omega,\mathscr{F},P)$ and $X \neq Y$ a.s.[P]. Then $||X-Y||_{L^\Phi} \neq 0$. Using the convexity of Φ,

$$\begin{aligned}\Phi(|X-Y|) &= \Phi(\varepsilon(\tfrac{1}{\varepsilon}\,|X-Y|) + (1-\varepsilon)(0)) \\ &\le \varepsilon\ \Phi(\tfrac{|X-Y|}{\varepsilon}) + (1-\varepsilon)\Phi(0) \\ &= \varepsilon\ \Phi(\tfrac{|X-Y|}{\varepsilon}).\end{aligned}$$

Integrating,

$$\int_\Omega \Phi(|X-Y|)dP \le \varepsilon \int_\Omega \Phi(\frac{|X-Y|}{\varepsilon})dP.$$

Now suppose $||X-Y||_{L^\Phi} \le 1$. Then

$$\begin{aligned}\int_\Omega \Phi(|X-Y|)dP &\le ||X-Y||_{L^\Phi} \int_\Omega \Phi(\frac{|X-Y|}{||X-Y||_{L^\Phi}})dP \\ &= ||X-Y||_{L^\Phi}\Phi(1),\end{aligned}$$

amply demonstrating the global uniform continuity of ξ.

We will not address the question of the local uniform convexity of ξ here. The hypothesis of Theorem 4 or possibly something similar might

guarantee the local uniform convexity of ξ.

V. A CONVERGENCE PRINCIPLE FOR A GENERALIZED CONDITIONAL EXPECTATION

Definition: Let $(\Omega,\mathscr{S},P)$ be a probability space, $\mathscr{F}$ be a sub σ-algebra of $\mathscr{S}$ and Φ,Ψ be an Orlicz pair. For $X\in L^{\Phi}(\Omega,\mathscr{S},P)$, we define $\hat{E}_{\Phi}(X|\mathscr{F})$ to be the unique solution to the variational problem

$$\min\left\{\int_{\Omega}\Phi(|X-Z|)dP:\ Z\in L^{\Phi}(\Omega,\mathscr{F},P)\right\}.$$

Recall from [5] that if $\Phi(x) = \frac{x^2}{2}$ for $x \geq 0$, then $\hat{E}_{\Phi}(\cdot|\mathscr{F})$ is ordinary conditional expectation given $\mathscr{F}$.

Remark: If the functional $X \overset{\xi}{\longmapsto} \int_{\Omega}\Phi(|X|)dP$ is locally uniformly convex then $\hat{E}_{\Phi}(\cdot|\mathscr{F})$ is norm continuous on $L^{\Phi}(\Omega,\mathscr{S},P)$.

Theorem 5: Let $(\Omega,\mathscr{S},P)$ be a probability space, $\{\mathscr{F}_n\}_{n=1}^{\infty}$ be an increasing family of sub σ-algebras of $\mathscr{S}$ and put $\mathscr{F} = \bigvee_{n=1}^{\infty}\mathscr{F}_n$. Suppose Φ,Ψ is an Orlicz pair so that $X \to \int_{\Omega}\Phi(|X|)dP$ is locally uniformly convex. Then for $X\in L^{\Phi}(\Omega,\mathscr{S},P)$,

$$\hat{E}_{\Phi}(X|\mathscr{F}_n) \xrightarrow{L^{\Phi}} \hat{E}_{\Phi}(X|\mathscr{F}) \text{ as } n\to\infty,$$

or equivalently,

$$\lim_{n\to\infty}\int_{\Omega}\Phi(|\hat{E}_{\Phi}(X|\mathscr{F}_n)-\hat{E}_{\Phi}(X|\mathscr{F})|)dP = 0.$$

Proof: This is an immediate consequence of Proposition 3. QED

We may now proceed as in [5] to get

Theorem 6: Let $(\Omega,\mathscr{S},P)$ be a probability space, $\{X(t): t\in[0,T]\}$ be a process on $(\Omega,\mathscr{S},P)$ taking values in a separable metric space M and Φ,Ψ be an Orlicz pair with $X \overset{\xi}{\longmapsto} \int_{\Omega}\Phi(|X|)dP$ locally uniformly convex. Then if $\{Q_n\}_{n=1}^{\infty}$ is a round off scheme for M and $\{P_m\}_{m=1}^{\infty}$ is an increasing sequence of partitions of $[0,T]$ whose meshes decrease to zero, we have for $Y\in L^{\Phi}(\Omega,\mathscr{S},P)$

$$\hat{E}_{\Phi}(Y|Q_n(X(t)):\ t\in P_m) \xrightarrow{L^{\Phi}} \hat{E}_{\Phi}(Y|X(t):\ t\in[0,T])$$

as $m,n\to\infty$, or equivalently,

$$\lim_{m,n\to\infty}\int_{\Omega}\Phi(|\hat{E}_{\Phi}(Y|Q_n(X(t)):\ t\in P_m) - \hat{E}_{\Phi}(Y|X(t):\ t\in[0,T])|)dP = 0.$$

VI. CONCLUSIONS AND A PARTING SHOT AT EARLIER WORK

The abstract principle developed in Section III subsumes the Banach space principle developed in [5]. Simply take the functional to be the norm.

We have enhanced the feasibility of studying the operators $\hat{E}_{\Phi}$ numerically in two ways. First, knowing that $\hat{E}_{\Phi}$ is continuous assures us that

it will tolerate small L^{Φ}-perturbations. Second, the result of Theorem 6 shows us we may approximate $\hat{E}_{\Phi}(Y|X(t): t\epsilon[0,T])$ by $\hat{E}_{\Phi}(Y|Q_n(X(t)): t\epsilon P_m)$. Since $\sigma(Q_n(X(t)): t\epsilon P_m)$ is finite, calculation of $\hat{E}_{\Phi}(Y|Q_n(X(t)): t\epsilon P_m)$ consists of solving a finite dimensional nonlinear optimization problem. A large body of knowledge now exists about the numerical solution of such problems.

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